Integration By Parts

\[ \int u\,dv = uv - \int v\,du \]

**Guidelines**

1. Try letting \( dv \) be the most complicated part of the integrand that fits a basic integration rule. Then \( u \) will be the remaining factor(s).

2. Try letting \( u \) be the portion of the integrand whose derivative is a function simpler than \( u \). Then \( dv \) will be the remaining factor(s).

Note that \( dv \) always includes the \( dx \) of the original integrand.

For \( \int x^n e^{ax} \, dx, \int x^n \sin(ax) \, dx, \int x^n \cos(ax) \, dx \) let \( u = x^n \)
\( dv = e^{ax} \, dx, \sin(ax) \, dx, \cos(ax) \, dx \)

For \( \int x^n \ln x \, dx, \int x^n \arcsin(ax) \, dx, \int x^n \arctan(ax) \, dx \) let \( u = \ln x, \arcsin(ax), \arctan(ax) \)
\( dv = x^n \, dx \)

For \( \int e^{ax} \sin(bx) \, dx, \int e^{ax} \cos(bx) \, dx \) let \( u = \sin(bx), \cos(bx) \)
\( dv = e^{ax} \, dx \)

The tabular method works well when \( u = x^n \)
Trig Integrals

Useful Identities:

\[ \int \sin^m x \cos^n x \, dx \quad \sin^2 x + \cos^2 x = 1 \]
\[ \sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \]

Guidelines

1. If \( m \) is odd and positive, let \( du = \sin x \, dx \). Convert the remaining sines into cosines using \( \sin^2 x = 1 - \cos^2 x \).
2. If \( n \) is odd and positive, let \( du = \cos x \, dx \). Convert the remaining cosines into sines using \( \cos^2 x = 2 - \sin^2 x \).
3. If \( m \) and \( n \) are even and positive, make repeated substitutions to create odd powers of cosines. Then see \#2.

\[ \sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \]

\[ \int \sec^m x \tan^n x \, dx \quad \text{Useful identity:} \quad \sec^2 x = 1 + \tan^2 x \]

Guidelines

1. If \( m \) is even and positive, let \( du = \sec^2 x \, dx \). Convert the remaining secants into tangents using \( \sec^2 x = 1 + \tan^2 x \).
2. If \( n \) is odd and positive, let \( du = \sec x \tan x \, dx \). Convert the remaining tangents into secants using \( \tan^2 x = \sec^2 x - 1 \).
3. If \( m = 0 \) and \( n \) is even and positive, factor out a \( \sec^2 x \) and substitute \( \sec^2 x = 1 + \tan^2 x \). Expand the remaining expression and repeat as needed.
4. If \( n = 0 \) and \( m \) is odd and positive, use integration by parts. \( dv = \sec^2 x \, dx \)
5. If none of the above, try converting to sines and cosines.

When sine and cosine have different angles, make one of the following substitutions:

- \( \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \)
- \( \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)] \)
- \( \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] \)
Trig Substitution

Look for radicals of form $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-a^2}$

1. For integrals with $\sqrt{a^2-u^2}$, let $u = a \sin \theta$.
   Then $\sqrt{a^2-u^2} = a \cos \theta$, where $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$

2. For integrals with $\sqrt{a^2+u^2}$, let $u = a \tan \theta$.
   Then $\sqrt{a^2+u^2} = a \sec \theta$, where $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$

3. For integrals with $\sqrt{u^2-a^2}$, let $u = a \sec \theta$.
   Then $\sqrt{u^2-a^2} = a \tan \theta$, where $0 \leq \theta \leq \frac{\pi}{2}$

Steps for Partial Fractions

1. Improper Fractions: If the degree of $N(x)$ is greater than the degree of $D(x)$, then $\frac{N(x)}{D(x)}$ is an improper fraction. Divide $N(x)$ by $D(x)$ to get $\frac{N(x)}{D(x)} = \text{polynomial} + \frac{N_1(x)}{D(x)}$

2. Factor Denominator: Completely factor the denominator into factors of the form $(px+q)^m$ and $(ax^2+bx+c)^n$ where $ax^2+bx+c$ is irreducible.

3. Linear Factors: For each factor of the form $(px+q)^m$, the sum of $m$ partial fractions is required.
   \[ \frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \ldots + \frac{A_m}{(px+q)^m} \]
   derivative of a line is a constant so a constant in each numerator

4. Quadratic Factors: For each factor of the form $(ax^2+bx+c)^n$, the sum of $n$ partial fractions is required.
   \[ \frac{A_{1x+B_1}}{ax^2+bx+c} + \frac{A_{2x+B_2}}{(ax^2+bx+c)^2} + \ldots + \frac{A_{nx+B_n}}{(ax^2+bx+c)^n} \]
   derivative of a quadratic is a line so a line in each numerator
To solve for constants:
1. Multiply the entire equation by the common denominator.
2. Plug in convenient values of \( x \), or the zeros of each linear factor, to solve for as many constants as you can.
3. Expanding each side and setting a polynomial = polynomial. Set the coefficients of like powers equal to create a system of equations. Solve for remaining constants.

**Improper Integrals**

1. If \( f \) is continuous on the interval \([a, \infty)\), then
   \[
   \int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx
   \]

2. If \( f \) is continuous on the interval \((-\infty, b]\), then
   \[
   \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx
   \]

3. If \( f \) is continuous on the interval \((-\infty, \infty)\), then
   \[
   \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx \quad \text{where } c \text{ is any real number.}
   \]

   For 1 and 2: the integral converges if the limit exists
   
   the integral diverges if the limit does not exists

   For 3: the integral on the left diverges if either of the integrals on the right diverges.

**L’Hôpital’s Rule:** If \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \), then
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}
\]
for any \( c \) in \((-\infty, \infty)\). Simplify and repeated as needed.